MEASURE AND INTEGRATION – REEXAMINATION SOLUTIONS
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1. Let $\Omega = \mathbb{N}$, $\mathcal{A} = P(\Omega)$ (the set of all subsets of Ω) and let p_n, $n \in \mathbb{N}$, be nonnegative numbers. Show that
$$\mu(A) = \sum_{n \in A} p_n, \quad A \subset \Omega,$$
defines a measure on Ω.
Solution. Since $\mu(A)$ is a sum of non-negative numbers, it is always non-negative (and possibly ∞). So μ is a function that assumes values in $[0,\infty]$. We have $\mu(\varnothing) = 0$ since the sum that defines μ is empty in this case. Let $A_n \in \mathcal{A}$, $n \in \mathbb{N}$ be pairwise disjoint sets. We have
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n \in \cup_{n=1}^{\infty} A_n} p_n = \sum_{n=1}^{\infty} \sum_{m \in A_n} p_m = \sum_{n=1}^{\infty} \mu(A_n)$$
(in the second equality, we have used the fact, proven in Analysis, that any ordering of a sum of non-negative numbers produces the same result).

2. Using the definition of Lebesgue measure, prove that for any Borel set $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we have
$$m(A) = m(A + x),$$
where
$$A + x = \{y + x : y \in A\}. \qquad (\clubsuit)$$
Solutions. Let $R_1, R_2, \ldots$ be rectangles covering $A + x$. Then, $-x + R_1, -x + R_2, \ldots$ are rectangles covering A, so
$$\sum \ell(R_n) = \sum \ell(-x + R_n) \geq m(A).$$
Taking the infimum over all covers of $A + x$, we obtain $m(A + x) \geq m(A)$. Using the same inequality with x replaced by $-x$ (noting that $A = (A + x) - x$), we obtain $m(A) \geq m(A + x)$.

3. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space such that $\{\omega\} \in \mathcal{A}$ for all $\omega \in \Omega$. Let $f : \Omega \to \mathbb{R}$ be a non-negative function and assume that there exists a countable set $E \subset \Omega$ such that $f(\omega) = 0$ for all $\omega \notin E$. Prove that f is measurable and that
$$\int_\Omega f \, d\mu = \sum_{x \in E} f(\omega)\mu(\{\omega\}).$$
Solutions. Any countable subset of Ω is in $\mathcal{A}$, so E and any subset of E are in $\mathcal{A}$. Fix $B \in \mathcal{B}(\mathbb{R})$; we have
$$f^{-1}(B) = \begin{cases} \{\omega \in E : f(\omega) \in B\} & \text{if } 0 \notin B; \\ \{\omega \in E : f(\omega) \in B\} \cup E^c & \text{if } 0 \in B. \end{cases}$$
That is, $f^{-1}(B)$ is either a subset of E or a subset of E together with E^c; in any case, $f^{-1}(B) \in \mathcal{A}$, so f is measurable.
Enumerate $E = \{d_1, d_2, \ldots\}$ (assuming here that E is infinite; the finite case is treated similarly). Then,
$$\int_\Omega f \, d\mu = \int_\Omega f \cdot \mathbb{1}_{\cup_{n=1}^\infty \{d_n\}} \, d\mu = \int_\Omega f \cdot \left(\sum_{n=1}^\infty \mathbb{1}_{\{d_n\}}\right) \, d\mu = \sum_{n=1}^\infty \int_\Omega f \, \mathbb{1}_{\{d_n\}} \, d\mu = \sum_{n=1}^\infty f(d_n) \cdot \mu(\{d_n\}),$$
where the next-to-last equality follows from the Monotone Convergence Theorem applied to series of functions (which is applicable here because f is positive).

4. (a) Let μ be a σ-finite measure on the measurable space $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ denotes the Borel σ-algebra on $\mathbb{R}$. Prove that $E = \{x \in \mathbb{R} : \mu(\{x\}) > 0\}$ is countable.

   (b) Let ν be another σ-finite measure on $(\mathbb{R}, \mathcal{B})$. Let

$$D = \{(x, x) : x \in \mathbb{R}\}$$

be the diagonal of $\mathbb{R}^2$. Show that

$$(\mu \otimes \nu)(D) = \sum_{x \in E} \mu(\{x\}) \cdot \nu(\{x\}).$$

Solutions.

(a) Since μ is σ-finite, there exists a sequence of measurable sets $\Omega_1 \subset \Omega_2 \subset \cdots$ with $\cup_k \Omega_k = \mathbb{R}$ and $\mu(\Omega_k) < \infty$ for all k. We have

$$\{x \in \mathbb{R} : \mu(\{x\}) > 0\} = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \{x \in \Omega_k : \mu(\{x\}) > 1/n\},$$

so it suffices to prove that for all n and k, the set $\{x \in \Omega_k : \mu(\{x\}) > 1/n\}$ is countable. To prove this, let $A \subset \{x \in \Omega_k : \mu(\{x\}) > 1/n\}$ be any countable set; then,

$$\infty > \mu(\Omega_k) \geq \mu(A) = \sum_{x \in A} \mu(\{x\}) \geq \frac{1}{n} \# A,$$

so A is finite. This proves that $\{x \in \Omega_k : \mu(\{x\}) > 1/n\}$ is finite.

(b)

$$(\mu \otimes \nu)(D) = \int_{\mathbb{R}} \mu(D^y)\, \nu(dy) = \int_{\mathbb{R}} \mu(\{y\})\, \nu(dy) = \int_E \mu(\{y\})\, \nu(dy),$$

where we have used part (a) and the previous exercise.

5. Let $A \subset \mathbb{R}^n$ be a compact (that is, closed and bounded) set and let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable function (with respect to Lebesgue measure). Prove that

$$\lim_{x \to \infty} \int_{A+x} f\, dm = 0,$$

where $A + x$ is as in (♣).

Solution. For $r > 0$, let $B_r = \{x \in \mathbb{R}^n : \|x\| \leq r\}$. The sequence of functions $f_n = |f| \cdot \mathbb{1}_{(B_n)^c}$ converges pointwise to 0, and the convergence is dominated by $|f|$, which is integrable. Hence, by the Dominated Convergence Theorem, $\int f_n\, dm \xrightarrow{n \to \infty} 0$. Now, given $\varepsilon > 0$, choose n such that $\int f_n\, dm < \varepsilon$, and then (using the fact that A is bounded) choose s such that, if $\|x\| \geq s$, then $A + x \subset (B_n)^c$. Then, if $\|x\| \geq s$,

$$\left| \int_{A+x} f\, dm \right| \leq \int_{A+x} |f|\, dm \leq \int_{(B_n)^c} |f|\, dm = \int_{\mathbb{R}^n} f_n\, dm < \varepsilon.$$

6. Let $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$ be integrable functions (with respect to Lebesgue measure). Assume that $f_n \to f$ uniformly and the limit $\displaystyle\lim_{n \to \infty} \int_{\mathbb{R}} f_n\, dm$ exists. Can we conclude that this limit is equal to $\int_{\mathbb{R}} f\, dm$? Prove or give a counterexample.

Solution. A counterexample is obtained by letting $f_n = \frac{1}{n} \cdot \mathbb{1}_{[0,n]}$, which converges uniformly to $f \equiv 0$, and $\lim \int f_n\, dm = 1 \neq 0 = \int f\, dm$.

7. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $1 \leq p < \infty$. Let $f_n : \Omega \to \mathbb{R}$ be a sequence of measurable functions converging pointwise to $f : \Omega \to \mathbb{R}$. Assume that $|f_n| \leq g$ for some $g \in \mathcal{L}^p(\Omega)$ and all n. Show that $f \in \mathcal{L}^p(\Omega)$ and that $\|f_n - f\|_p \to 0$ as $n \to \infty$.
Solution. We have $f_n^p \to f^p$ pointwise and, for all n, $|f_n^p| \leq |g^p|$. Note that, since g is assumed to belong to $\mathcal{L}^p(\Omega)$, we have $\int_\Omega |g|^p \, d\mu < \infty$, that is, $|g^p|$ is integrable. Hence, the Dominated Convergence Theorem implies that f^p is integrable, so $f \in \mathcal{L}^p(\Omega)$.

Now note that $|f_n - f| \to 0$ pointwise and, since $|f| \leq g$,

$$|f_n - f|^p \leq (|f_n| + |f|)^p \leq (2|g|)^p$$

and the right-hand side is integrable as already observed. Again using the Dominated Convergence Theorem, we conclude that $\int_\Omega |f_n - f|^p \, d\mu \to 0$, so $\|f_n - f\|_p \to 0$.